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Hyperbolic Complex Yang–Baxter Equation and Hyperbolic Complex Multiparametric Quantum Groups

Wu Ya-Bo[1] **and Zhong Zai-Zhe**[1]

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The hyperbolic complex Yang–Baxter equation is equivalent to a system consisting of two ordinary Yang–Baxter equations, and a hyperbolic complex quantum group is isomorphic to a direct product of two quantum groups. As a concrete example, the quantum group $GL_H(\Gamma;\ \xi_{ij})$ with hyperbolic complex multiparameter is isomorphic to a direct product of two quantum groups $GL(X;\ q_{ij})$ and $GL(Y;\ r_{ij})$ with ordinary multiparameter.

1. INTRODUCTION AND PRELIMINARIES

Hyperbolic complex structures have interesting applications in physics (Hucks, 1993; Wu, 1992, 1994). Recently, quantum group theory (Drinfeld, 1986; Manin, 1988; Reshetikhin *et al.*, 1989) has made great progress, but the hyperbolic complex structure is seldom involved in this field. Zhong (1992) has proved that the hyperbolic complex quantum groups $SU_q(2)$ and $GL_q(2)$ are isomorphic to a real quantum group and a direct product of two real quantum groups, respectively. This leads us to expect that there should be more complete and thorough results if we introduce the hyperbolic complex structure into quantum group theory. In fact, we have discovered an interesting fact: the ordinary pure imaginary unit i ($i^2 = -1$) does not appear in discussions of most quantum groups. Therefore, we may deduce that introducing the hyperbolic complex structure does not change the basic framework of quantum groups in most cases. In this paper, we prove that a hyperbolic complex solution of the Yang–Baxter equation (YBE) can be constructed from two arbitrary real solutions of YBE, and vice versa. Furthermore, we

[1]Department of Physics, Liaoning Normal University, Dalian, Liaoning, 116029, China.

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also prove that an arbitrary hyperbolic complex quantum group is isomorphic to a direct product of two real quantum groups. These results are just an extension of Zhong's (1992) results.

Let ϵ denote the pure hyperbolic imaginary unit, i.e., $\epsilon^2 = +1$, $\epsilon \neq +1$. If f and g are real numbers, then the set of all hyperbolic complex numbers $f + \epsilon g$ forms a commutative ring H (Yaglom, 1988), according to addition and multiplication. If A is an algebra over K and $R^1 \subset K$, the hyperbolic complexification of A is written as A_H, the element of which takes the form $a + \epsilon b$, $a, b \in A$. The A_H is an algebra over H. For any A_H we have the identity

$$\prod_{i=1}^{n} \left\{\frac{1}{2}(X_i + Y_i) + \frac{\epsilon}{2}(X_i - Y_i)\right\} = \frac{1}{2}\left(\prod_{i=1}^{n} X_i + \prod_{i=1}^{n} Y_i\right) + \frac{\epsilon}{2}\left(\prod_{i=1}^{n} X_i - \prod_{i=1}^{n} Y_i\right) \tag{1}$$

This identity plays an important role in the following discussion.

2. THE HYPERBOLIC COMPLEX YANG–BAXTER EQUATION AND ITS SOLUTIONS

In order to stress the role of the parameter q, in the following a quantum Yang–Baxter matrix is written as $\check{R}(q)$. Suppose that p and q are two real parameters, and the $n \times n$ matrices $\check{R}(p)$ and $\check{\tilde{R}}(q)$, respectively, are the solutions of the YBEs

$$\check{R}_{12}(p)\check{R}_{23}(p)\check{R}_{12}(p) = \check{R}_{23}(p)\check{R}_{12}(p)\check{R}_{23}(p) \tag{2}$$

$$\check{\tilde{R}}_{12}(q)\check{\tilde{R}}_{23}(q)\check{\tilde{R}}_{12}(q) = \check{\tilde{R}}_{23}(q)\check{\tilde{R}}_{12}(q)\check{\tilde{R}}_{23}(q)$$

Let

$$\check{R}_H(p, q) = \frac{1}{2}[\check{R}(p) + \check{\tilde{R}}(q)] + \frac{\epsilon}{2}[\check{R}(p) - \check{\tilde{R}}(q)] \tag{3}$$

Then by use of equation (1) we obtain

$$\begin{aligned}
&\check{R}_{H12}(p, q)\check{R}_{H23}(p, q)\check{R}_{H12}(p, q) \\
&= \frac{1}{2}[\check{R}_{12}(p)\check{R}_{23}(p)\check{R}_{12}(p) + \check{\tilde{R}}_{12}(q)\check{\tilde{R}}_{23}(q)\check{\tilde{R}}_{12}(q)] \\
&\quad + \frac{\epsilon}{2}[\check{R}_{12}(p)\check{R}_{23}(p)\check{R}_{12}(p) - \check{\tilde{R}}_{12}(q)\check{\tilde{R}}_{23}(q)\check{\tilde{R}}_{12}(q)] \\
&= \frac{1}{2}[\check{R}_{23}(p)\check{R}_{12}(p)\check{R}_{23}(p) + \check{\tilde{R}}_{23}(q)\check{\tilde{R}}_{12}(q)\check{\tilde{R}}_{23}(q)]
\end{aligned}$$

$$+ \frac{\epsilon}{2} [\check{R}_{23}(p)\check{R}_{12}(p)\check{R}_{23}(p) - \check{\tilde{R}}_{23}(q)\check{\tilde{R}}_{12}(q)\check{\tilde{R}}_{23}(q)]$$

$$= \check{R}_{H23}(p, q)\check{R}_{H12}(p, q)\check{R}_{H23}(p, q) \tag{4}$$

This means that $\check{R}_H(p, q)$ is a solution of a YBE

$$\check{R}_{H12}\check{R}_{H23}\check{R}_{H12} = \check{R}_{H23}\check{R}_{H12}\check{R}_{H23} \tag{5}$$

where, however, the $\check{R}$ matrix is hyperbolic complex.

Notice that the above matrix $\check{R}_H$ has two independent real deformation parameters p and q. For $\check{R}$ matrices related to quantum groups generally the form of an entry is

$$[\check{R}(p)]^{ij}_{kl} = C^{ij}_{kl} + pD^{ij}_{kl} + \frac{1}{p} E^{ij}_{kl} \tag{6}$$

where C^{ij}_{kl}, D^{ij}_{kl}, and E^{ij}_{kl} are constants. Therefore, if $\check{R}(p)$ and $\check{\tilde{R}}(q)$ take the same form, i.e., $\check{\tilde{R}}(q) = \check{R}(p \to q)$ (the arrow denotes the substitution), then by a simple calculation according to equation (3) we obtain

$$[\check{R}_H]^{ij}_{kl} = C^{ij}_{kl} + \xi D^{ij}_{kl} + \frac{1}{\xi} E^{ij}_{kl}$$

$$\xi = \frac{1}{2}(p + q) + \frac{\epsilon}{2}(p - q) \tag{7}$$

In addition, we notice that so long as p and q both are not zero, then

$$\xi^{-1} = \frac{1}{\xi} = (pq)^{-1}\left[\frac{1}{2}(p + q) - \frac{\epsilon}{2}(p - q)\right]$$

always exists. This means that if the entries of the $\check{R}$ matrices take the form of equation (6), then the hyperbolic complex $\check{R}_H$ has a hyperbolic complex deformation parameter $\xi = \alpha + \epsilon\beta$, with $\alpha = \frac{1}{2}(p + q)$ and $\beta = \frac{1}{2}(p - q)$, and $\check{R}_H(\xi)$ has the same form as a real $\check{R}(p)$ matrix, i.e., $\check{R}_H(\xi) = \check{R}(p \to \xi)$.

Conversely, if $\check{R}_H$ is a YBE solution with two independent real parameters such as α and β, then $\check{R}_H$ can be surely decomposed into two ordinary YBE solutions. In fact, the entries of $\check{R}_H$ can be written as

$$[\check{R}_H(\alpha, \beta)]^{ij}_{kl} = F^{ij}_{kl}(\alpha, \beta) + \epsilon G^{ij}_{kl}(\alpha, \beta) \tag{8}$$

where F and G do not contain ϵ. Let

$$A = F + G, \qquad B = F - G \tag{9}$$

Then

$$\check{R}_H = \frac{1}{2}[A(\alpha, \beta) + B(\alpha, \beta)] + \frac{\epsilon}{2}[A(\alpha, \beta) - B(\alpha, \beta)] \quad (10)$$

By using equation (1) again, from the real and imaginary parts of equation (10), we see that

$$\begin{aligned} A_{12}(\alpha, \beta)A_{23}(\alpha, \beta)A_{12}(\alpha, \beta) &= A_{23}(\alpha, \beta)A_{12}(\alpha, \beta)A_{23}(\alpha, \beta) \\ B_{12}(\alpha, \beta)B_{23}(\alpha, \beta)B_{12}(\alpha, \beta) &= B_{23}(\alpha, \beta)B_{12}(\alpha, \beta)B_{23}(\alpha, \beta) \end{aligned} \quad (11)$$

This means that A and B are both real Yang–Baxter matrices with the real deformation parameters α and β.

In the case that the entries of $\check{R}_H$ take the form of equation (6), then the deformation parameter of the $\check{R}_H$ matrix is a hyperbolic complex number $\xi = \alpha + \epsilon\beta$. In this case, the real matrices A and B, which are obtained by decomposing $\check{R}_H$, have real parameters $p = \alpha + \beta$ and $q = \alpha - \beta$, respectively, i.e.,

$$\begin{aligned} \check{R}_H(\xi) &= \frac{1}{2}[A(p) + B(q)] + \frac{\epsilon}{2}[A(p) - B(q)] \\ \xi &= \alpha + \epsilon\beta = \frac{1}{2}(p + q) + \frac{\epsilon}{2}(p - q) \end{aligned} \quad (12)$$

To sum up, we see that the set of hyperbolic complex Yang–Baxter matrices $\check{R}_H$ corresponds one-to-one to a pair of two real matrices $\check{R}$, $\check{\tilde{R}}$,

$$\check{R}_H \rightleftarrows (\check{R}, \check{\tilde{R}}) \quad (13)$$

3. HYPERBOLIC COMPLEX QUANTUM GROUPS AND ISOMORPHIC RELATIONS

The classification of the solutions of the YBE is the same as that of the quantum envelope algebras of classical Lie algebras; therefore, for a given solution $\check{R}(p, q)$ of the YBE we may suppose that there is a corresponding quantum group $G(p, q)$. A quantum matrix $M(p, q) \in G(p, q)$ must satisfy the Yang–Baxter relation

$$\begin{aligned} &(M(p, q) \otimes 1)(1 \otimes M(p, q))\check{R}(p, q) \\ &= \check{R}(p, q)(M(p, q) \otimes 1)(1 \otimes M(p, q)) \end{aligned} \quad (14)$$

In the following, an asterisk denotes hyperbolic complex conjugation, i.e., $(a + \epsilon b)^* = a - \epsilon b$, where a and b are real numbers. A quantum matrix $M(p, q) = (M^i_j(p, q))$ is called real if $(M^i_j(p, q))^* = M^i_j(p, q)$ under the operator *. If $M(p, q)$ and $\tilde{M}(r, s)$ are two real quantum matrices obeying equation (14) in $G(p, q)$ and $G(r, s)$, respectively, then we obtain a hyperbolic complex quantum matrix

$$M_H = (M^i_{Hj})$$

$$M^i_{Hj} = \frac{1}{2}\,[M^i_j(p, q) + \tilde{M}^i_j(r, s)] + \frac{\epsilon}{2}\,[M^i_j(p, q) - \tilde{M}^i_j(r, s)] \tag{15}$$

Notice that here M_H has four deformation parameters p, q, r, and s. By using equations (1) and (14), we can directly prove

$$(M_H \otimes 1)(1 \otimes M_H)\check{R}_H = \check{R}_H(M_H \otimes 1)(1 \otimes M_H) \tag{16}$$

Thus, we obtain the map

$$\rho\colon \quad G(p, q) \times G(r, s) \to G_H(p, q; r, s)$$
$$\rho(M(p, q), M(r, s)) = M_H(p, q; r, s) \tag{17}$$

It is easily proved that the following results hold for ρ:

$$\rho[(M, \tilde{M})\cdot(M', \tilde{M}')] = \rho(M, \tilde{M})\cdot\rho(M', \tilde{M}')$$
$$\rho(1, 1) = 1$$
$$\rho(M^{-1}, \tilde{M}^{-1}) = [\rho(M, \tilde{M})]^{-1} \tag{18}$$

This means ρ is an isomorphism,

$$\rho\colon \quad G(p, q) \times G(r, s) \approx G_H(p, q; r, s) \tag{19}$$

In particular, if $\check{R}_H$ has hyperbolic complex deformation parameters ξ and η, then so does the corresponding quantum group $M_H \in G_H$,

$$M_H = M_H(\xi, \eta)$$
$$\xi = \frac{1}{2}(p + r) + \frac{\epsilon}{2}(p - r)$$
$$\eta = \frac{1}{2}(q + s) + \frac{\epsilon}{2}(q - s) \tag{20}$$

In this special case, equation (18) becomes

$$\rho\colon \quad G(p, q) \times G(r, s) \approx G_H(\xi, \eta)$$
$$\xi = \frac{1}{2}(p + r) + \frac{\epsilon}{2}(p - r)$$
$$\eta = \frac{1}{2}(q + s) + \frac{\epsilon}{2}(q - s) \tag{21}$$

Therefore, we have obtained a general extension of Zhong's (1992) result.

The above discussion may be generalized to the case of multiparametric quantum groups. As an example, we consider the quantum group $GL(X; q_{ij})$. The corresponding Yang–Baxter matrix is $\check{R}(X; q_{ij})$ (Schrrimacher, 1991),

$$(\check{R}(X; q_{ij}))^{ij}_{kl} = \delta^i_k \delta^j_l \left(\delta^{ij} + \theta^{ji} \frac{1}{q_{ij}} + \theta^{ij} \frac{q_{ij}}{X} \right) + \delta^i_l \delta^j_k \theta^{ij} \left(1 - \frac{1}{X} \right) \quad (22)$$

where θ^{ij} is equal to 1 for $i > j$ and 0 otherwise. Therefore we obtain a hyperbolic complex Yang–Baxter matrix

$$\check{R}(\Gamma; \xi_{ij}) = \frac{1}{2} [\check{R}(X; q_{ij}) + \check{R}(Y; r_{ij})] + \frac{\epsilon}{2} [\check{R}(X; q_{ij}) - \check{R}(Y; r_{ij})] \quad (23)$$

We easily see that the concrete form of the entries of $\check{R}(\Gamma; \xi_{ij})$ is just the same as that of $\check{R}(X; q_{ij})$, and its deformation parameters are hyperbolic complex, i.e.,

$$(\check{R}(\Gamma; \xi_{ij}))^{ij}_{kl} = \delta^i_k \delta^j_l \left(\delta^{ij} + \theta^{ji} \frac{1}{\xi_{ij}} + \theta^{ij} \frac{\xi_{ij}}{\Gamma} \right) + \delta^i_l \delta^j_k \theta^{ij} \left(1 - \frac{1}{\Gamma} \right)$$

$$\Gamma = \frac{1}{2} (X + Y) + \frac{\epsilon}{2} (X - Y)$$

$$\xi_{ij} = \frac{1}{2} (q_{ij} + r_{ij}) + \frac{\epsilon}{2} (q_{ij} - r_{ij}) \quad (24)$$

which is in accord with the results discussed above. Therefore we have

$$GL(X; q_{ij}) \times GL(Y; r_{ij}) \approx GL_H(\Gamma; \xi_{ij}) \quad (25)$$

By virtue of the relations

$$\Gamma^{-1}(x, y) = \Gamma(x^{-1}, y^{-1}), \qquad \xi^{-1}_{ij}(q_{ij}, r_{ij}) = \xi(q^{-1}_{ij}, r^{-1}_{ij}) \quad (26)$$

and the results of Corrigan *et al.* (1990), we have the following property of the YBE solutions:

$$\check{R}(\Gamma^{-1}; \xi^{-1}_{ij}) = [\check{R}(\Gamma; \xi_{ij})]^{-1} \quad (27)$$

etc.

We discover that there is a special subgroup in the quantum group $GL_H(\Gamma; \xi_{ij})$, i.e., if X, Y, q_{ij}, and r_{ij} all approach a single parameter q, then Γ, $\xi_{ij} \to q$ also, and equation (25) becomes

$$GL_q(n) \times GL_q(n) \approx GL_q(n; H) \subset GL_H(\Gamma; \xi_{ij}) \quad (28)$$

which is just the result given by Zhong (1992).

It is easily seen that the above discussion can be used for other quantum groups, and there are similar results. To sum up, the concrete example shows that a hyperbolic complex quantum group with hyperbolic complex multiparameter usually is isomorphic to a direct product of two quantum groups with ordinary multiparameter. This is an extension of results concerning ordinary Lie groups (Zhong, 1985, 1992).

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